Exponential Error Bounds and Decoding Complexity for Block Codes Constructed by Trellis Codes

Shigeichi Hirasawa, Masao Kasahara

Abstract

In our previous studies, we have discussed on block concatenated codes from random coding arguments. Block codes constructed by trellis codes are used for inner codes to reduce the decoding complexity of the over-all concatenated codes. In this paper, we focused upon only the block codes constructed by the trellis codes. Error exponents and decoding complexity for terminated trellis codes and generalized tail biting (GTB) random trellis codes, and their relationships are derived, where the GTB trellis codes consist of full tail biting (FTB) trellis codes, partial tail biting (PTB) trellis codes and direct truncated (DT) trellis codes. We show that the PTB trellis codes at all rates except for low rates are superior among the GTB trellis codes, in a sense that they have smaller upper bound on the probability of decoding error for given decoding complexity.

Keywords: Trellis codes, Error exponent, Decoding complexity, Terminated trellis codes, Generalized tail biting trellis codes

Introduction

In this decade, we have studied on trellis codes [11] for reducing the decoding complexity of the concatenated codes [5], [6], [7]. The block concatenated codes [2] and their generalized version [8] with trellis inner codes have been discussed from random coding arguments. The trellis codes are effectively used as the inner codes so that we can obtain larger exponents of the concatenated codes without increasing the over-all decoding complexity.

In this paper, we focused upon only the block codes constructed by the trellis codes. Error exponents and decoding complexity for terminated trellis codes and generalized tail biting (GTB) trellis codes are discussed. The terminated trellis

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原稿受付日：2010 年 10 月 4 日
原稿受理日：2011 年 2 月 10 日

codes [3] are simple deriving methods for the block codes with respectable expense in rates. While the GTB trellis codes [9] are known to be one of the most powerful codes for converting trellis codes into the block codes with no loss in rates. The GTB trellis codes consist of full tail biting (FTB) trellis codes, partial tail biting (PTB) trellis codes and direct truncated (DT) trellis codes. Since the FTB trellis codes require an intolerable increase in decoding complexity, much efforts have been devoted to the studies on suboptimum decoding algorithms for the FTB trellis codes [1], [9] or efficient maximum likelihood decoding algorithms for the GTB trellis codes [12]. Unfortunately, however, the decoding complexity of the latter algorithms in worst cases is the same as that of the complete maximum likelihood decoding algorithm, although it is asymptotically the same as that of the Viterbi algorithm when the signal to noise ratio becomes large.

On the other hand, we assume the using of complete maximum likelihood decoding of the GTB trellis codes, since we are interested in random coding arguments. The q-ary GTB trellis code can be constructed as follows [9] : Let the encoder be initialized by the last part $v'(\leq v)$ symbols of the information symbols of length u, where v is the constraint length of the trellis code, and ignore the output of the encoder corresponding to vinformation symbols. Next input all u information symbols into the encoder yielding b channel symbols per information symbol, and output the codeword of length $N = ub$, where rate r is defined by $r = \dfrac{1}{b} \ln q$. We then have a (ub, u) block code over GF (q). Note that the case of $v' = v$ gives the FTB trellis code, and that of $v' = 0$, the DT trellis code. We have analyzed error exponents and the decoding complexity for the FTB random trellis code. There is a possibility such that the probability of decoding error for the GTB trellis codes is smaller than that of ordinary block codes with the same decoding complexity, even if complete maximum likelihood decoding of the GTB trellis codes is performed.

First, we briefly review results obtained for the ordinary block codes and the terminated trellis codes. Second, we derive the error exponents and the decoding complexity for the GTB trellis codes. We show that the PTB trellis codes at all rates except for low rates are superior among the GTB trellis codes, in a sense that they have smaller upper bound on the probability of decoding error for given decoding complexity. Next, the DT trellis codes are compared with the terminated trellis codes. Finally, we compare the performance of the terminated trellis codes and the GTB trellis codes (which consist of the FTB, the PTB, and the DT trellis codes).

Throughout this paper, assuming a discrete memoryless channel with capacity C, we discuss the lower bound on the reliability function (usually called the error exponent) and asymptotic decoding complexity measured by the computational work [10].

In Section 2, we briefly review the performance of the ordinary block codes and the terminated trellis codes. In Section 3, the error exponents and the decoding complexity for the GTB trellis codes are derived. The term $o(1)$s are disregarded, since we are interested in an asymptotic behavior. Section 4 describes conclusions and further works.

2 Preliminaries

2.1 Block codes

Let an (N, K) block code over GF (q) be a code of length N, number of information symbols K and rate R, where

$$R = \frac{K}{N} \ln q \quad (K \leq N). \text{ [nats/symbol]} \tag{1}$$

From random coding arguments for an ordinary block code, we have the following lemma.

Lemma 1 (**Ordinary block codes** [4]) There exists a block code of length N and rate R for which the probability of decoding error $\Pr(\mathcal{E})$ and the decoding complexity G satisfy

$$\Pr(\mathcal{E}) \leq \exp[-NE(R)] \quad (0 \leq R \leq C), \tag{2}$$

and

$$G \sim \exp[NR], \tag{3}$$

where $E(\cdot)$ is (the lower bound on) the block code exponent [3], and the symbol "$\sim$" indicates asymptotic equality[1]. □

Table 1 Asymptotic results on error exponents and decoding complexity for block codes

Block code	Error exponent	Decoding complexity G	Upper bound on $\Pr(\cdot)$
Ordinary block code (Lemma 1)	$E(R)$	$\exp[NR]$	$G^{-\frac{E(R)}{R}}$
Terminated trellis code (Lemma 2)	$E(R)$ [3]	q^{v}	$G^{-\frac{1-\theta}{\theta}\frac{E(R)}{R}}$
GTB trellis code (Theorems 1 and 2)			
DT trellis code ($\theta' = 0$)	$E(r)$ [3]	q^{v}	$G^{-\frac{1}{\theta}\frac{E(r)}{r}}$
PTB trellis code ($0 < \theta' < \theta$)	$e_G(r)$	$q^{v+v'}$	$G^{-\frac{1}{\theta+\theta'}\frac{e_G(r)}{r}}$ [2]
FTB trellis code ($\theta' = \theta$)	$e_G(r)$	q^{2v}	$G^{-\frac{1}{2\theta}\frac{e_G(r)}{r}}$ †

2.2 Trellis codes

Let a (u, v, b) trellis code over GF (q) be a code of branch length u, branch constraint length v, yielding b channel symbols per branch and rate r which satisfies

$$r = \frac{1}{b} \ln q. \quad [\text{nats/symbol}] \tag{4}$$

Hereafter, we denote $\frac{v}{u}$ by a parameter θ, i.e.,

$$\theta = \frac{v}{u} \quad (0 < \theta \leq 1). \tag{5}$$

Letting

$$N = ub, \tag{6}$$

and

$$R = \frac{(u-v)r}{u} = (1-\theta)r, \tag{7}$$

we have the following lemma:

Lemma 2 (**Terminated trellis codes** [3]) There exists a block code of length N and rate R obtained by a (u, v, b) terminated random trellis code for which the equations (2) and (8) are satisfied, where N and R are given by (6) and (7), respectively.

$$G \sim q^{v}. \tag{8}$$

□

For ordinary block code and for terminated trellis codes, results derived are shown in Table 1 (See Appendix A).

Note that the following equation holds between $E(R)$ and $e(r)$, where $e(\cdot)$ is (the lower bound on) the trellis code exponent [3]:

$$E(R) = \max_{0 < \mu \leq 1} (1-\mu) e\left(\frac{R}{\mu}\right), \tag{9}$$

which is called the concatenation construction [3].

3 Generalized tail biting trellis codes

The GTB trellis code is introduced as a powerful converting method for maintaining a larger error exponent with no loss in rates, although the decoding complexity increases. The GTB trellis codes can be constructed as follows [9]: Suppose an encoder of a (u, v, b) trellis code. First, initialize the encoder by inputting the last v' information (branch) symbols of u information (branch) symbols, and ignore the output of the encoder. Next, input all u information symbols into the encoder, and output the codeword of length $N = ub$ in channel symbols. As the result, we have a (ub, u) block code of rate $r = \frac{1}{b} \ln q$ over GF (q) by the tail biting method. Hereafter we denote $\frac{v'}{u}$ by a parameter bu θ', i.e.,

$$\theta' = \frac{v'}{u} \quad (0 \leq \theta' \leq \theta \leq 1). \tag{10}$$

The GTB trellis codes are composed of :

(i) Direct truncated (DT) trellis codes for $v' = 0$ ($\theta' = 0$);

(ii) Partial tail biting (PTB) trellis codes for $0 < v' < v$ ($0 < \theta' < \theta \leq 1$); and

(iii) Full tail biting (FTB) trellis codes $v' = v$ ($\theta' = \theta \leq 1$).

3.1 Exponential error bounds for GTB trellis codes

Theorem 1 There exists a block code of length N and rate r obtained by a GTB random trellis code with $0 \leq \theta' \leq \theta \leq 1$ for which the probability of decoding error $\Pr(\mathcal{E})$ satisfies

$$\Pr(\mathcal{E}) \leq \exp[-Ne_G(r)] \ (0 \leq r < C), \tag{11}$$

where

$$e_G(r) = \min\{\theta e(r), E[(1-\theta')r], E(\theta' r)\} \quad (0 \leq \theta' \leq \theta \leq 1). \tag{12}$$

Proof: Let $\boldsymbol{w}$ be a message sequence of (branch) length u, where all messages are generated with the equal probability. Rewrite the sequence $\boldsymbol{w}$ as

$$\boldsymbol{w} = (\boldsymbol{w}_{u-v}, \boldsymbol{w}_v), \tag{13}$$

where $\boldsymbol{w}_{u-v}$ is the former part of $\boldsymbol{w}$ (length $u-v$), and $\boldsymbol{w}_v = (\boldsymbol{w}_{v'}, \boldsymbol{w}_{v-v'})$ the latter part of $\boldsymbol{w}$ (length v). First initialize the encoder by inputting $(\boldsymbol{w}_{v'}, 0^{v-v'})$, where 0^m is all 0 sequence of length m. Next input $\boldsymbol{w}$ into the encoder. Then output the coded sequence $\boldsymbol{x}$ of length[3] $N = ub$. Suppose the $q^{v'}$ Viterbi trellis diagrams, each of which starts at the state $s_i (i = 1, 2, \cdots, q^{v'})$ depending on $\boldsymbol{w}_{v'}$, and ends at the state $s_j \in \mathcal{S}_i$, where the number of the states s_j, i.e., $|\mathcal{S}_i|$, is $q^{v-v'}$ [9] (See Figure 1). The Viterbi decoder generates the maximum likelihood path $\boldsymbol{w}^{(i)}$ in the trellis diagram for starting at s_i and ending at $s_j \in \mathcal{S}_i$. Computing $\max_i \boldsymbol{w}^{(i)} = \hat{\boldsymbol{w}}$, the decoder outputs $\hat{\boldsymbol{w}}$.

Starting states	Ending states	Starting states	Ending states
$s_1 = (0, 0)$,	$\mathcal{S}_1 = \{(0, 0)\}$	$s_1 = (0, 0)$,	$\mathcal{S}_1 = \{(0, 0), (0, 1)\}$
$s_2 = (0, 1)$,	$\mathcal{S}_2 = \{(0, 1)\}$		
$s_3 = (1, 0)$,	$\mathcal{S}_3 = \{(1, 0)\}$	$s_3 = (1, 0)$,	$\mathcal{S}_3 = \{(1, 0), (1, 1)\}$
$s_4 = (1, 1)$,	$\mathcal{S}_4 = \{(1, 1)\}$		
(a) FTB trellis code for $v = v' = 2$		(b) PTB trellis code for $v = 2$ and $v' = 1$	

Figure 1 Examples of FTB and PTB trellis codes

The decoding error occurs when $\{\boldsymbol{w} \neq \hat{\boldsymbol{w}}\}$. Without loss of generality, let the true path be $\boldsymbol{w} = 0^u$ which starts at s_1 (and ends at s_1). We then have three types of decoding error, i.e., $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ (See Figure 2).

The probability of decoding error $\Pr(\mathcal{E}_1)$ within the trellis diagram starting at s_1 (and ending at s_1) for a (u, v, b) random trellis code is given by [3]

$$\begin{aligned}\Pr(\mathcal{E}_1) &\leq K_1 N \exp[-vbE_0(\rho)] \quad (0 \leq \rho \leq 1)\\ &= \exp\{-N\theta[e(r)-o(1)]\}, \qquad (14)\end{aligned}$$

where an error event begins at any time and $o(1) = \dfrac{\ln K_1 N}{N\theta} \to 0$ as $N \to \infty$.

The probability of decoding error $\Pr(\mathcal{E}_2)$ within the trellis diagram starting at s_1 and ending at $s_j \in \mathcal{S}_1$ $(j \neq 1)$ is given by [3]

$$\begin{aligned}\Pr(\mathcal{E}_2) &\leq q^{(v-v')\rho} \exp\{-vbE_0(\rho)\}\\ &= \exp\{-N[E_0(\rho)-\rho(1-\theta')r]\}\\ &= \exp\{-NE[(1-\theta')r]\}, \qquad (15)\end{aligned}$$

since the number of the possible adversaries is $q^{v-v'}-1$.

While the probability of decoding error $\Pr(\mathcal{E}_3)$ within trellis diagrams starting at $s_i (i \neq 1, i = 2, 3, \cdots, q^{v'})$ and ending at $s_j \in \mathcal{S}_i$ is also given by

$$\begin{aligned}\Pr(\mathcal{E}_3) &\leq q^{v'\rho} \exp\{-vbE_0(\rho)\}\\ &= \exp\{-N[E_0(\rho)-\rho\theta' r]\}\\ &= \exp\{-NE[(\theta' r]\}, \qquad (16)\end{aligned}$$

since the number of the possible adversaries is $q^{v'}-1$.

From (14), (15) and (16), the probability of over-all decoding error $\Pr(\mathcal{E})$ is bounded by the union bound:

$$\begin{aligned}\Pr(\mathcal{E}) &\leq \Pr(\mathcal{E}_1) + \Pr(\mathcal{E}_2) + \Pr(\mathcal{E}_3)\\ &= \exp\{-N[e_G(r)-o(1)]\}, \qquad (17)\end{aligned}$$

where $e_G(r)$ is given by (12) and $o(1) = \dfrac{\ln K_1 N}{N\theta} + \dfrac{\ln 3}{N} \to 0$ as $N \to \infty$. Disregard of the term $o(1)$ in (17) gives (12). □

Note that in the proof of Theorem 1, not only the trellis code construction but the block code construction appear as shown in Figure 2.

In our previous paper [5] , we have discussed the FTB trellis codes, where we restricted ourselves to be $0 < \theta \leq \dfrac{1}{2}$, since larger error exponents which are dominated by $e(\cdot)$ are obtained and the 2 decoding complexity for the block concatenated code with the FTB trellis inner codes is remained to that of the original concatenated codes.

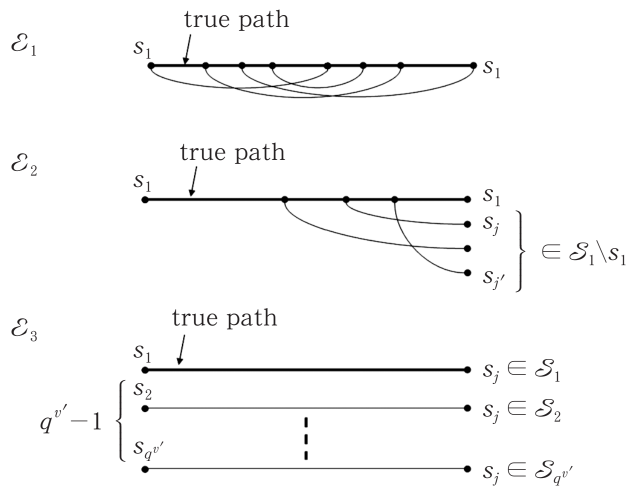

Figure 2　Three types of error events

Collorary 1　(FTB trellis codes [5]) The probability of decoding error $\Pr(\mathcal{E})$ for the FTB trellis codes with $0 < \theta \leq \frac{1}{2}$ satisfies

$$\Pr(\mathcal{E}) \leq \exp[-N\theta e(r)] \quad (0 \leq r < C), \tag{18}$$

and the decoding complexity G for the FTB trellis codes is given by

$$G \sim q^{2v} = \exp[2N\theta r]. \tag{19}$$

□

Comparison between the DT trellis codes and terminated trellis codes leads the following corollary.

Collorary 2　The DT trellis codes have a smaller upper bound on the probability of decoding error than both the terminated trellis codes and the ordinary block codes at the same decoding complexity.
Proof: See Appendix B. □

Example 1　On a very noisy channel[4], the exponents for the GTB trellis codes are depicted in Figure 3(a), (b) and Figure 4. We see that:

(a) The largest exponent is obtained at $\theta' = 0.5$ for $\frac{1}{2} < \theta \leq 1$, since $E[(1-\theta')r] = E(\theta' r)$ holds, hence the PTB trellis codes are the best among GTB trellis codes from the view-point of error exponents.

(b) While the largest exponent is obtained at $\theta = \theta'$ for $0 < \theta \leq \frac{1}{2}$, hence the FTB trellis codes are the best among the GTB trellis codes.

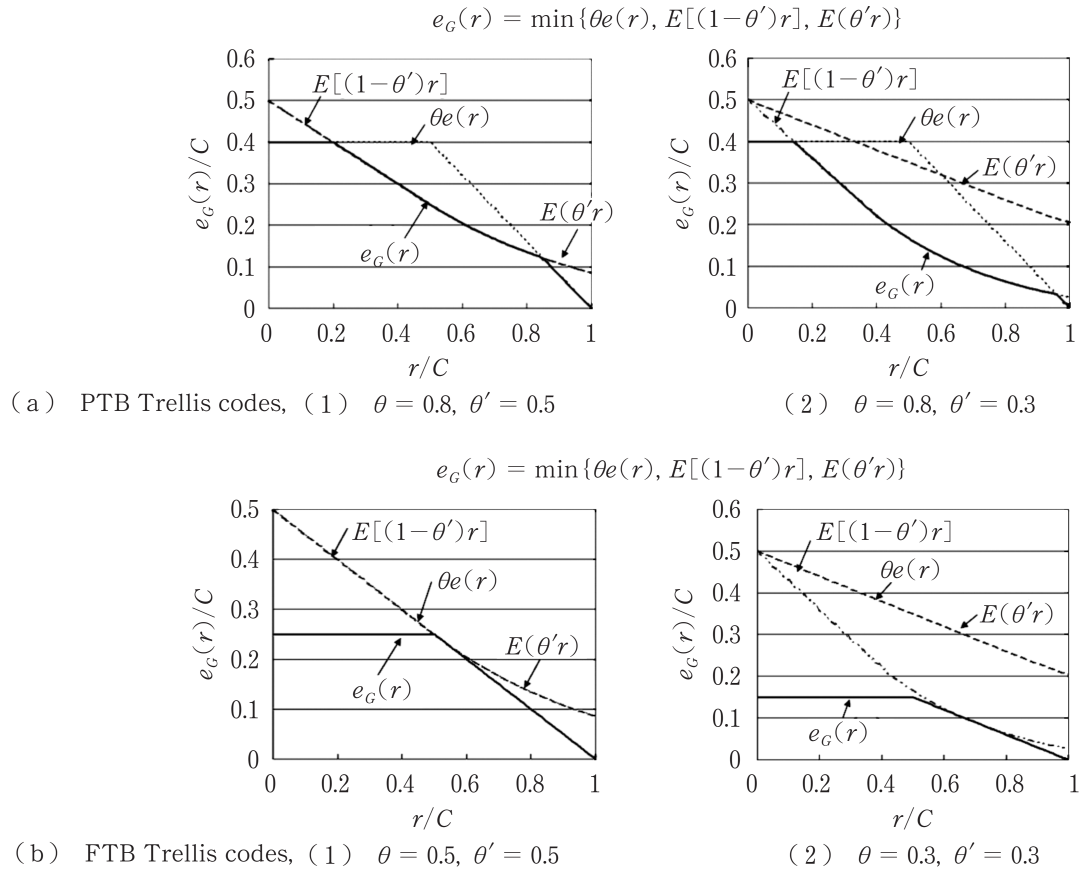

Figure 3 Example of construction of error exponents for GTB trellis codes over a very noisy channel

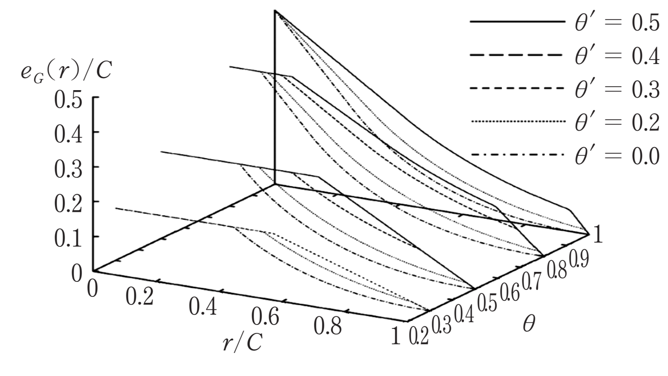

Figure 4 Exponents $g_G(r)$ of G for a very noisy channel.

3.2 Decoding complexity for GTB trellis codes

The maximum likelihood decoder for the Viterbi algorithm requires $u^2 q^{v+1}$ comparisons (See derivavtions in [5] Appendix A) for each trellis diagram and performs them in parallel for $q^{v'}$ trellis diagrams for the GTB trellis codes. We then have Theorem 2, where $u^2 q^{v+v'+1} = u^2 q q^{v+v'} = \exp\{Nr[\theta+\theta'+o(1)]\}$ ($o(1) = \dfrac{2\ln u + \ln q}{vbr} \to 0$ as $v \to \infty$) and $q = \exp[br]$ are used.

Theorem 2 The decoding complexity G of the GTB trellis code is given by

$$G \sim q^{v+v'} = \exp[N(\theta+\theta')r] \quad (0 \leq \theta' \leq \theta \leq 1). \tag{20}$$

□

The results derived in Theorem 1 and Theorem 2 are also shown in Table 1.

3.3 Upper bound on probability of decoding error for same decoding complexity

Next, we evaluate the probability of decoding error $\Pr(\mathcal{E})$ by taking into account the decoding complexity G so that coding methods can be clearly compared [3]. Let us assume that the code length N and rate $R = r$ are the same for all conversion methods. To rewrite $\Pr(\mathcal{E})$ in terms of G for the ordinary block codes, we have $G \sim \exp[NR]$ from (3), i.e.,

$$N \sim \frac{1}{R} \ln G. \tag{21}$$

We then have [3]

$$\Pr(\mathcal{E}) \leq G^{-\frac{E(R)}{R}}. \tag{22}$$

Since (20) holds for the GTB trellis code, we have the following corollary.

Collorary 3 For the GTB trellis code, we have

$$\Pr(\mathcal{E}) \leq G^{-g_G(r)}, \tag{23}$$

where

$$g_G(r) = \frac{1}{(\theta+\theta')r} \min\{\theta e(r), E[(1-\theta')r], E(\theta' r)\}, \tag{24}$$

and the term $g_G(r)$ in an exponent part of G is taken to be minimum for $0 \leq \theta' \leq \theta \leq 1$.

Proof: See Appendix C. □

A similar derivation gives the evaluations for the DT trellis code and for terminated trellis code as shown in Table 1, where $q^v = \exp[vbr] = \exp[N\theta r]$ holds (See Appendix A).

Example 2 On a very noisy channel, the exponents $g_G(r)$ of G in (24) for the GTB trellis code are shown in Figure 5. We see that:

(a) For $\frac{1}{2} < \theta \leq 1, g_G(r)$ is the largest at $\theta' = 0.5$, hence the PTB trellis codes are the best among the GTB trellis codes at all rates except for low rates. The DT and the FTB trellis codes have smaller values of $g_G(r)$ than the PTB trellis codes.

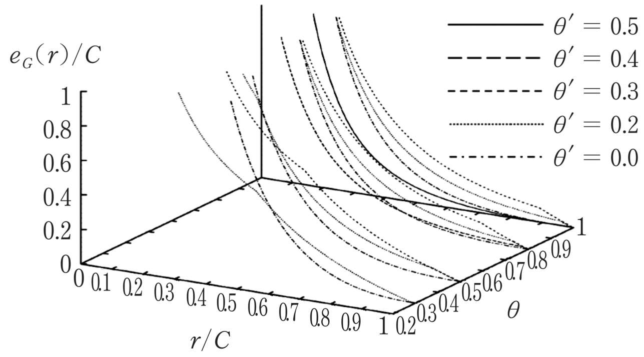

Figure 5 Error exponents $e_G(r)$ of GTB trellis codes for a very noisy channel.

(b) While for $0 < \theta \leq \frac{1}{2}$, $g_G(r)$ is the largest at $\theta = \theta'$, hence the FTB trellis codes are the best among the GBT codes, and the DT trellis codes have smaller values of $g_G(r)$ than the FTB trellis codes.

4 Concluding remarks

We have derived the error exponents and the decoding complexity for block codes converted from the trellis codes. We have shown that the PTB trellis codes at all rates except for low rates are superior among the GTB trellis codes, in a sense that they have the smallest upper bound on the probability of decoding error for given decoding complexity. This result suggests us that we can attain high performance by the PTB trellis codes with a careful choice of the parameter θ' for given θ. Detail discussions [7] are omitted here, it has been also clarified that the DT trellis inner codes are effective among the GTB trellis codes for constructing the generalized version of concatenated codes, Codes $C^{(J)}$, to keep the same decoding complexity as the original concatenated codes. If we can allow increasing the decoding complexity, larger exponents are obtained by Codes $C^{(J)}$ with the GTB trellis inner codes. We also show that larger error exponents are obtained by the generalized version of concatenated codes, if the decoding complexity is allowed to be larger than that of the original concatenated code, although it is still in polynomial order.

A detailed analysis on upper bounds on the probability of decoding error for the GTB trellis codes with different parameters θ and θ' at the same decoding complexity will be in further investigation. Although the random coding arguments suggest some useful aspects to construct the code, we should note to make them applicable to a practical code, which is also a future work.

Acknowledgement
One of the present authors, S. H. would like to thank Professor M. Kobayashi at Shonan Institute of Technology, and Dr. G. Hosoya at Waseda University for their helpful work to write this paper.

Notes

1 Strictly speaking, $G \sim N^2 \exp[NR]$ holds, since likelihood comparisons between two codewords require N logical operations and N shift operations, and the maximum number of comparisons of codewords is $\exp[NR]$. We have used $N^2 \exp[NR] = \exp NR[1+o(1)]$, $o(1) = \frac{2 \ln N}{NR} \to 0$ as $N \to \infty$, where the term $o(1)$ is ignored in (3).

2 Terms in an exponent part of G are taken to be minimum for given $0 \le \theta' \le \theta \le 1$.

3 Note that GTB random trellis coding requires every channel symbol on every branch be chosen independently at random with the probability $\boldsymbol{p}$ which maximizes $E_0(\rho, \boldsymbol{p})$ on nonpathological channels [3].

4 On a very noisy channel, the upper bound and the lower bound to the error exponent are approximately the same for all rates, hence it is called the true error exponent. The error exponents of orthogonal codes over the unlimited bandwidth white Gaussian channel coincide with those of codes over a very noisy channel [2].

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Appendix A: Derivations of error exponents and decoding complexity for a terminated trellis code in Table 1.

For a terminated trellis code, we have

$$\begin{aligned}\Pr(\mathcal{E}) &\le (u-v)K_1 \exp\{-vb[e(r)-o(1)]\}\\ &\le NK_1 \exp\{-N\theta[e(r)-o(1)]\}\\ &\le \exp\{-N[E(R)-o'(1)]\},\end{aligned} \tag{25}$$

where

$$E(R) = \max_{0\le\rho\le1}[E_0(\rho)-\rho R] \qquad (R=(1-\theta)r), \tag{26}$$

and K_1 is a constant independent of u, and $E_0(\rho)$ is the Gallger's function. Substituting $\theta = 1-\mu$ in (9) and disregarding $o'(1)$ in (25), we have an error exponent $E(R)$. Obviously, the decoder requires q comparisons at each node for each step, where the number of nodes is q^v, and repeats them u steps. Since these operations are carried out by u logic units, we have u^2q^{v+1} computational work as the decoding complexity in total. We have used $u^2q^{v+1} = u^2q\exp[vbr] = \exp\{vbr[1+o(1)]\}$, $o'(1) = \dfrac{2\ln u + \ln q}{vbr} \to 0$, as $v\to\infty$, where the term $o'(1)$ is ignored in Table 1.

Appendix B: Proof of Corollary 2

From Table 1, we see that

$$\frac{E(R)}{R} \le \frac{(1-\theta)E(R)}{\theta R} \le \frac{E(r)}{\theta r} \quad \left(0<\theta\le\frac{1}{2}\right), \tag{27}$$

and

$$\frac{(1-\theta)E(R)}{\theta R} \le \frac{E(R)}{R} \le \frac{E(r)}{\theta r} \quad \left(\frac{1}{2}<\theta\le 1\right), \tag{28}$$

completing the proof.

Appendix C: Proof of Corollary 3

Substitution of (20) into (11) gives

$$\begin{aligned}\Pr(\mathcal{E}) &\le \exp[-Ne_G(r)]\\ &= G^{-\frac{Ne_G(r)}{N(\theta+\theta')r}},\end{aligned} \tag{29}$$

after a little manipulation.

トレリス符号から構成される
ブロック符号の誤り指数と復号計算量

平澤　茂一，笠原　正雄

筆者らはトレリス符号で構成されるブロック符号について解析し，ブロック連接符号の内部符号として用いることを提案しその有効性を示した。ここでは一連の研究成果から，トレリス符号の部分のみを取り上げ，ランダム符号化と最尤復号を仮定し，誤り指数と復号計算量の立場から理論的に解明した結果を報告する。

トレリス符号はその構造上復号計算量を低減し，同時に誤り指数を向上させる可能性がある。トレリス符号の優れた性質を保持したままブロック符号化する方法には，終端法と一般化テールバイティング（GTB）法がある。前者は良く知られた方法で，既に古くから解析されている。ここでは後者について詳しく論じている。後者は符号のパラメータを選ぶことにより，フルテールバイティング（FTB）法・部分テールバイティング（PTB）法・直接切断（DT）法に分けられる。GTB 法を統一的に評価し，PTB 法が与えられた復号計算量の下に大きな誤り指数が得られることを示す。

なお，誤り指数は復号誤り確率の上界を与える関数で，同一の符号長では誤り指数が大きいほど小さな復号誤り確率の上界を与える。また，トレリス符号をブロック符号化することにより復号遅延時間を一定の値に抑えることが可能となり，実用上望ましい性質を持たせることが出来る。

キーワード： トレリス符号，誤り指数，復号計算量，終端されたトレリス符号，一般化テールバイティングトレリス符号